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Equivalence between isospectrality and isolength spectrality for a certain class of planar billiard domains

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Abstract

Isospectrality of the planar domains which are obtained by successive unfolding of a fundamental building block is studied in relation to isolength spectrality of the corresponding domains. Although an explicit and exact trace formula such as Poisson's summation formula or Selberg's trace formula is not known to exist for such planar domains, equivalence between isospectrality and isolength spectrality in a certain setting can be proved by employing the matrix representation of 'transplantation of eigenfunctions'. As an application of the equivalence, transplantable pairs of domains, which are all isospectral pair of planar domains and therefore counter-examples of Kac's question 'can one hear the shape of a drum?', are numerically enumerated, and it is found that, at least up to the domain composed of 13 building blocks, transplantable pairs coincide with those constructed by the method due to Sunada.

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1. Introduction

A famous question of Kac, 'can one hear the shape of a drum?', is concerned with isospectrality of the planar domains [1]. The bounded domains D_1 and D_2 are called isospectral if two domains have the same eigenvalue spectrum up to the degree of multiplicities, i.e. $\mathrm{spec}(D_1) = \mathrm{spec}(D_2)$, where

$$\mathrm{spec}(D) = \{\mu_1 \leqslant \mu_2 \leqslant \cdots \mid \Delta f_i = \mu_i f_i \text{ in } D,\ f_i = 0 \text{ on } \partial D\ \ (i = 1, 2, \ldots)\}. \tag{1}$$

Kac's question is alternatively stated as 'are the planar domains D_1 and D_2 congruent if $\mathrm{spec}(D_1) = \mathrm{spec}(D_2)$?'.

Isolength spectrality of the domains is analogously introduced for the length spectrum of closed billiard trajectories on the corresponding domains D_1 and D_2. Also for isolength spectrality one can pose the same question as Kac. The problem we discuss here is equivalence between isospectrality and isolength spectrality on the planar domains of a certain class.

If either the eigenvalue spectrum or the length spectrum determines the shape of the domain uniquely, the equivalence problem becomes obvious since a congruent pair of domains trivially provides both the isospectrum and isolength spectrum. However, it is known that there exists a non-trivial case where equivalence between isospectrality and isolength spectrality is concluded. Such a situation was first reported by Milnor [2]. It is known that there exist the self-dual lattices L_1 and L_2 in $\boldsymbol{R}^{16}$ which are not congruent in the sense that no rotation of $\boldsymbol{R}^{16}$ carries L_1 to L_2, nevertheless they own the same length spectrum. That is, the quotients of $\boldsymbol{R}^{16}$ by these lattices, $\boldsymbol{R}^{16}/L_1$ and $\boldsymbol{R}^{16}/L_2$, are non-isometric but isolength spectral. Furthermore the isospectrality can also be derived from the exact trace formula which represents the duality relation between the spectrum of the Laplacian acting on the flat torus and the corresponding length spectrum of closed geodesics. For the flat torus which is defined as a quotient $\boldsymbol{R}^n/\Gamma$ of $\boldsymbol{R}^n$ by a lattice Γ of rank n, the set of eigenvalues of Helmholtz equation $\Delta f = \mu f$ is explicitly given as

$$\{4\pi^2\|\sigma\|^2 \mid \sigma \in \Gamma^*\}. \tag{2}$$

Here Γ^* denotes the dual lattice of Γ, i.e. $\Gamma^* = \{\sigma \in \boldsymbol{R}^n \mid \sigma \cdot \gamma \in \boldsymbol{Z}\ \forall \gamma \in \Gamma\}$. The corresponding length spectrum, which is the set of closed geodesics on the same torus, is expressed just by the distance between the origin and each lattice point on Γ:

$$\{\|\gamma\| \mid \gamma \in \Gamma\}. \tag{3}$$

The two spectra are connected via Poisson's summation formula or Jacobi identity:

$$\sum_{\sigma\in\Gamma^*} \exp(-4\pi^2\|\sigma\|^2 t) = (4\pi t)^{-n/2}\, \mathrm{vol}\,(\boldsymbol{R}^n/\Gamma) \sum_{\gamma\in\Gamma} \exp(-\|\gamma\|^2/4t). \tag{4}$$

This trace formula immediately leads us to equivalence between the isospectrality and isolength spectrality, that is, two flat tori are isospectral if and only if they have the same length spectrum. Milnor's work is not a direct answer to the original version of Kac's question because it is not concerned with the planar domains, but from Milnor's example one learns that there really exist non-congruent domains with a common length and eigenvalue spectrum.

In more general cases where two Riemannian manifolds have a common finite Riemannian covering, a sufficient condition for isospectrality was given by Sunada [3]. Let G be a finite group which acts freely on a certain compact Riemannian manifold M by isometries. Sunada's theorem tells us that the *quotients M/H_1 and M/H_2 are isospectral if there exist two subgroups H_1 and H_2 of G, which satisfy the so-called almost conjugate condition*:

$$^\sharp([g] \cap H_1) = ^\sharp([g] \cap H_2) \tag{5}$$

for any conjugacy class $[g]$ in G. The proof is based on a sort of trace formula, which can be regarded as a prototype of the Selberg trace formula [3, 4]. If subgroups H_1 and H_2 are conjugate in the usual sense, then these quotients M/H_1 and M/H_2 are merely isometric. However it had been known that there exist triplets (G, H_1, H_2) such that H_1 and H_2 are almost conjugate but not conjugate in G. Such triplets were used to construct some isospectral but non-isometric pairs of Riemannian surfaces by Buser [5] and Brooks and Tse [6]. Sunada also proved that M/H_1 and M/H_2 have the same length spectrum of closed geodesics if the condition (5) is fulfilled [3]. Therefore, in this case, isolength spectrality is derived together with isospectrality from the same source.

In the case of planar domains, after a quarter of a century, Gordon *et al* [7] also solved Kac's original question negatively by constructing an isospectral but non-congruent pair of planar domains (see figure 1). For the construction, they used a version of Sunada's theorem, which

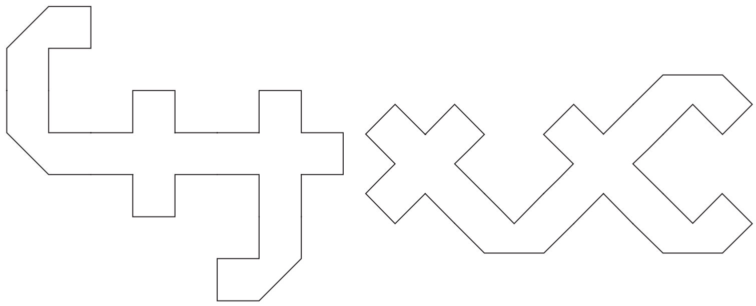

Figure 1. The isospectral pair of domains constructed by Gordon *et al*.

was extended to the orbiford (billiard) setting by Bérard [8]. Since the source of isospectrality was Sunada's theorem, isolength spectrality naturally follows as well.

However, if there exists an isospectral pair of planar domains which are not given by Sunada's construction, equivalence of isospectrality and isolength spectrality cannot straightforwardly be discussed. In fact, no one can exclude such a situation because the Sunada condition gives merely a sufficient condition for the isospectrality and also isolength spectrality. Nevertheless, as we will show below, it is possible to relate them through the notion of so-called *transplantation of eigenfunctions*. The transplantation method was originally invented by Buser *et al* [9] to reexamine isospectrality of some given domains. They constructed 17 families of isospectral pairs of domains based on the Sunada method, and confirmed their isospectrality by the transplantation method. As far as planar domains are concerned, these pairs are, to the authors' knowledge, all isospectral pairs ever known.

It should be remarked that the transplantation method is not independent of the Sunada theorem. In the above case of quotient manifolds, Brooks gave an alternative proof of Sunada's theorem based on the transplantation method, which appears to be a necessary condition for the assumption of the theorem [10]. However equivalence between these two methods has not been proved in the case of planar domains.

The organization of this paper is as follows. We first explain the transplantation method in section 2 and give its matrix representation in section 3. In section 4, we provide a proof showing that isospectrality and isolength spectrality is equivalent for a certain class of domains, that is those domains constructed by successive unfolding of a fundamental building block. All isospectral pairs ever known belong to this class. In section 5 we enumerate isospectral pairs of domains and compare them with the table obtained by Buser *et al* [9]. It is shown that our labour in searching for isospectral pairs is considerably reduced as a by-product of our result in section 4. Section 6 is devoted to a summary and discussion of our work.

2. Transplantation method

Let us consider two domains D_1 and D_2 as in figure 2. We here describe the transplantation method by taking an example of the proof of isospectrality based on it. We follow the exposition given by Buser *et al* [9].

Transplantation is a procedure which cuts an arbitrary function f defined on D_1 into its restrictions on each building block, say $f_1, f_2, \ldots, f_7$, and rebuilds a new function g on D_2 by superposing $f_1, f_2, \ldots, f_7$. Note that such a procedure relies crucially on the peculiarity of the domains, which are composed of several pieces obtained by successive reflection of a certain common building block.

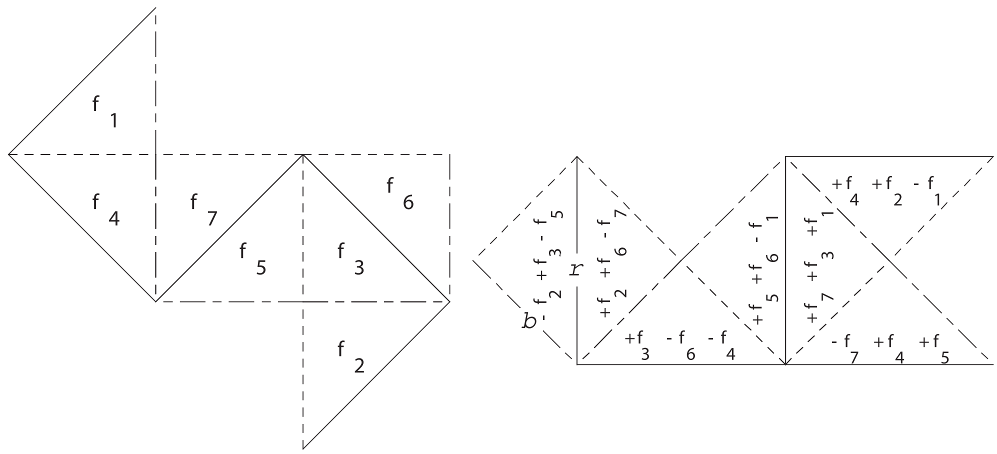

Figure 2. An isospectral pair of domains and the transplantation of eigenfunctions.

By the transplantation shown in figure 2, any eigenfunction f with any eigenvalue μ on D_1 is transplanted onto D_2 so that the transform g is also an eigenfunction with the same eigenvalue μ. This can actually be confirmed by checking the following conditions:

- $\Delta g = \mu g$ *in the interior of each piece.*
  This is always true because g is given as a sum of several f_i on each piece.
- *The function g is smoothly connected on all reflecting segments.*
  This is not trivial, but true in this example. On the reflecting segment r in figure 2, for example, $-f_2 + f_3 - f_5$ fits smoothly with $+f_2 + f_6 - f_7$ because f_3 and f_6, or $-f_5$ and $-f_7$, were smoothly connected on the original domain D_1 and $-f_2$ and f_2 also fit smoothly on the segment by the reflection principle. We can also check the smoothness condition on the other reflecting segments in the same way.
- *The function g vanishes on the boundary of* D_2.
  This is not trivial, but true in this example. On the boundary segment b in figure 2, for example, $-f_2 + f_3 - f_5$ vanishes because $-f_2$ cancels f_3 there and f_5 was already zero. We can also check the boundary condition on the other boundary segments in the same way.

One more condition is invertibility of transplantation, which guarantees that the dimension of the space belonging to the eigenvalue μ, i.e. the multiplicity of μ, on D_1 is equal to the dimension on D_2 or less. This is expressed as the condition,

- *The transplantation is invertible as a linear map.*
  This is not trivial, but can be easily checked to be true for this example.

Thus these four conditions lead to $\mathrm{Spec}(D_1) \subset \mathrm{Spec}(D_2)$. The last condition becomes the check to see that any eigenfunction with any eigenvalue ν on D_2 can be transplanted onto D_1 conversely; that is, $\mathrm{spec}(D_2) \subset \mathrm{spec}(D_1)$. In this way, we know that these two domains are isospectral.

These are conditions for eigenfunctions satisfying Dirichlet boundary conditions, but they are also Neumann isospectral. This can be verified by changing all the minus signs before each element of the transplanted function f_i to plus signs when one superposes them on the transformed domain. Conversely it is possible to construct Dirichlet isospectral pairs once

Table 1. The correspondence between the unfolding rule and the edge-coloured graph.

Unfolding rule		Edge-coloured graph
Pieces	$\longleftrightarrow$	Vertices
Reflecting segments	$\longleftrightarrow$	Edges
Kind of segment	$\longleftrightarrow$	Colour of edge

Neumann isospectra are at hand. The procedure is as follows: (i) for both domains, attach the code $\sigma_i = 1$ or -1 to each building block i such that the codes of neighbouring blocks are different and (ii) let the building block i with code σ_i be transformed onto the building block i' with code σ_i', and if $\sigma_i \sigma_{i'} = 1$ then the sign before f_i is set to be plus and otherwise minus. It is easy to show that the resulting pair of domains are Dirichlet isospectral. Therefore, isospectrality with respect to the Dirichlet and Neumann boundary conditions is equivalent.

We also remark that the above proof is independent of the choice of building block. This means that this example represents not only a single pair of isospectral domains but also a family of isospectral pairs, which are realized by replacing a building triangle simultaneously with a different shaped building block. In particular, we can obtain the example of Gordon *et al* in figure 1 by replacing the building block in figure 2 by a heptagon.

3. Transplantability and its matrix representation

In this section, we provide an alternative and more transparent condition for the *transplantability* of eigenfunctions. In what follows, we limit ourselves to the class of domains constructed by successive reflection of a fundamental building block, which we call *unfolded domains*, since transplantation of eigenfunctions is most simply performed between such domains. In order to make successive unfolding possible, each building block is supposed to have three line segments at which the building block is reflected. The domains shown in figures 1 and 2 are the examples. Hereafter we denote by N the number of building blocks composing the unfolded domain.

The whole shape of the unfolded domain is determined by the choice of the building block and the rule specifying unfolding. The unfolding rule can be identified with the edge-coloured graph and the correspondence is given in table 1. Each piece obtained by reflection is represented by the vertex of the graph, and the reflecting segment by its edge. The three boundary segments of the building block are distinguished by the colour of the edges. Some boundary segments of the building block form the boundary of the whole unfolded domain, and we represent the edge corresponding to one such boundary segment as the closed loop. An example of an edge-coloured graph is demonstrated in figure 3.

The structure of a graph is also represented by the adjacency matrix. That is, the reflecting rule to generate an unfolded domain P is given by the triplet of adjacency matrices M_a^P, M_b^P and M_c^P defined as follows:

$$\left(M_\gamma^P\right)_{ij} = \begin{cases} 1 & \text{if the piece } i \text{ is adjacent to } j \text{ by segment } \gamma \\ 0 & \text{otherwise.} \end{cases} \qquad \gamma = a, b, c. \tag{6}$$

Diagonal element $\left(M_\gamma^P\right)_{ii}$ is set to be unity if the piece i has the boundary segment γ. The adjacency matrix is, in general, not uniquely determined even if one fixes the graph because there remains a freedom to determine the order of labelling vertices. In this setting, two unfolded domains P and Q are congruent if there exists a permutation matrix U such that

$$U M_a^P U^{-1} = M_a^Q \qquad U M_b^P U^{-1} = M_b^Q \qquad U M_c^P U^{-1} = M_c^Q. \tag{7}$$

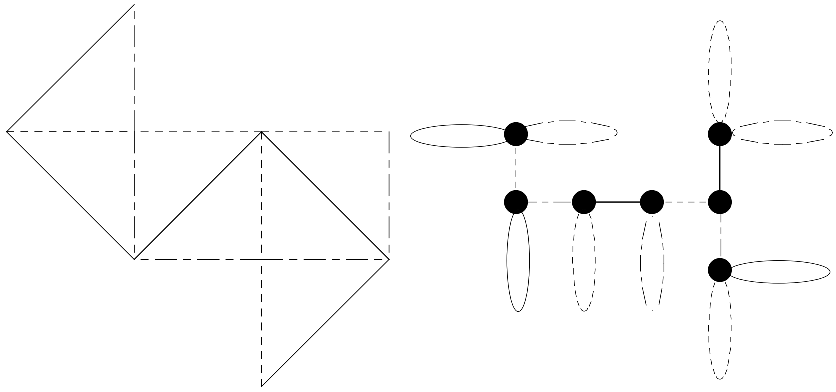

Figure 3. An example of an edge-coloured graph representing an unfolding rule. The three types of curve (solid, dotted, broken with dots) denote three colours.

Here the permutation matrix is defined as an $N \times N$ matrix in which each row or column has only one unity, and all the other elements are zero. If the building block has some symmetry, for example reflection symmetry with respect to a certain axis, it might happen that a non-permutation matrix connects the congruent pair P and Q. However, if one destroys the symmetry by changing the shape of the building block, such an 'accidental' situation can be avoided. The condition (7) for a permutation matrix U, therefore, becomes the necessary and sufficient condition for the congruency of P and Q in this sense.

Next we give the condition for transplantability of eigenfunctions of two given unfolded domains using the adjacency matrices. By transplantability we mean the possibility that the isospectrality of a pair of unfolded domains can be verified by the transplantation method.

The transplantation procedure described in section 3 is explicitly expressed as follows. Let P and Q be two unfolded domains composed of N copies of a common building block. Let $T : L^2(P) \ni f \mapsto g \in L^2(Q)$ be a linear transformation. We say that the transformation T is a *transplantation* if the transform g is obtained by cutting and pasting f, namely,

$$g_i = \sum_{j=1}^{N} T_{ij} f_j \qquad i = 1, 2, \ldots, N \tag{8}$$

where f_j and g_i denote restrictions of f and $g = Tf$ to each piece, respectively. Also as mentioned in section 2, it is not trivial at all that the transform g becomes an eigenfunction on the transformed domain at this stage.

For given unfolded domains P and Q, we say that they are *transplantable* if there exists an invertible transplantation T satisfying the following conditions:

$$T M_a^P T^{-1} = M_a^Q \qquad T M_b^P T^{-1} = M_b^Q \qquad T M_c^P T^{-1} = M_c^Q. \tag{9}$$

It can be easily checked that smoothness on all the reflecting segments and the boundary conditions on the unfolded domains are satisfied if the above conditions are fulfilled. Here, the existence of the inverse matrix T^{-1} corresponds to the invertible condition. The condition (9) is the one for transplantability of Neumann boundary condition, but, as remarked in section 2, it is equivalent to transplantability of the Dirichlet case. As seen in the previous section, P and Q are isospectral if they are transplantable.

Note that if T is a permutation matrix, then P and Q are merely congruent. Therefore, in order to construct an isospectral but non-congruent pair of unfolded domains, we must find a matrix satisfying the condition (9), but not a permutation matrix.

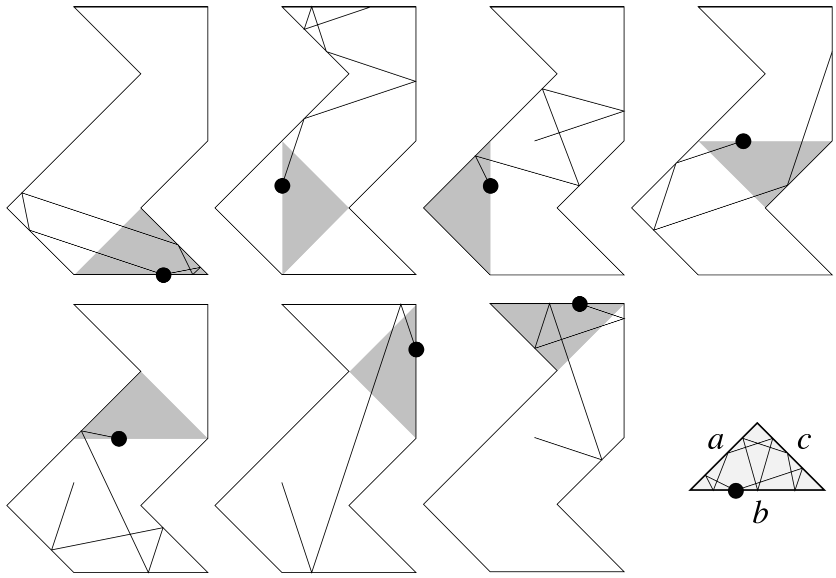

Figure 4. Lifts of a closed billiard trajectory o with the sequence $\gamma = babacbacbc$. For the sequence γ, $n^P(\gamma)$ is equal to 1.

4. Transplantability and isolength spectrality

We next consider the length spectrum of the unfolded domain. The length spectrum is the set of lengths of closed trajectories or periodic orbits on the billiard domain. We first consider the relation between closed trajectories on the unfolded domain P and its fundamental building block B. Note that any periodic orbit on P can be regarded as the lift of a closed trajectory on B, because its projection is always a periodic orbit on B. However, the converse is not necessarily true. A single closed orbit on B yields N trajectories as its lift on the unfolded domain P, which are sometimes closed orbits on P but the others not. By the construction of the adjacency matrix M_γ^P, it is easy to see that the number of closed lifts of a given closed trajectory on B is counted as

$$n^P(\gamma) = \mathrm{Tr}(M_{\gamma_m}^P M_{\gamma_{m-1}}^P \cdots M_{\gamma_1}^P) \tag{10}$$

where $\gamma = \gamma_1\gamma_2\cdots\gamma_m$ ($\gamma_i = a, b, c$) denotes the sequence representing the order in which a given closed trajectory on B hits the boundary segments (for example, see figure 4). Such a sequence is not determined uniquely for a given closed orbit. In fact, if $\gamma = \gamma_1\gamma_2\cdots\gamma_m$ is a sequence specifying some closed trajectory, then all sequences obtained by its cyclic shift such as $\gamma_2\cdots\gamma_m\gamma_1$ and $\gamma_3\cdots\gamma_m\gamma_1\gamma_2$ can be regarded as sequences giving the same closed trajectory. However the number of closed lifts is invariant because of the relation

$$\begin{aligned}
\mathrm{Tr}(M_{\gamma_m}^P \cdots M_{\gamma_2}^P M_{\gamma_1}^P) &= \mathrm{Tr}(M_{\gamma_1}^P M_{\gamma_m}^P \cdots M_{\gamma_2}^P M_{\gamma_1}^P (M_{\gamma_1}^P)^{-1}) \\
&= \mathrm{Tr}(M_{\gamma_1}^P M_{\gamma_m}^P \cdots M_{\gamma_2}^P) \\
&= \cdots .
\end{aligned} \tag{11}$$

Thus the length spectrum of closed trajectories for P is determined by the length spectrum for B and $n^P(\gamma)$.

In order that two unfolded domains P and Q constructed by the same-shaped building block B have the same length spectrum, it is sufficient that the numbers of closed lifts should coincide for all possible sequences of γ. It can easily be checked that $n^P(\gamma)$ and $n^Q(\gamma)$ are invariant if P and Q are transplantable, i.e. the condition (9) is satisfied for the adjacent matrices of P and Q. Therefore we obtain the following claim.

Proposition 1. *Let P and Q be two unfolded domains obtained by N times successive reflections of the same building block. If P and Q are transplantable, then $n^P(\gamma) = n^Q(\gamma)$ for any sequence γ, which leads to isolength spectrality of P and Q.*

The inverse of the above proposition is to inquire whether or not $n^P(\gamma) = n^Q(\gamma)$ gives the sufficient condition for transplantability. The following statement gives the answer: the condition $n^P(\gamma) = n^Q(\gamma)$ is precisely the analogue of the transplantability of eigenfunctions.

Proposition 2. *Let P and Q be two unfolded domains obtained by N times successive reflection of the same building block. If $n^P(\gamma) = n^Q(\gamma)$ for any sequence γ, then P and Q are transplantable, which leads to isospectrality of P and Q.*

Proof. We have shown that transplantability implies isospectrality in section 3. Here we will show that transplantability follows from $n^P(\gamma) = n^Q(\gamma)$. Let G^P and G^Q be the groups generated by adjacency matrices:

$$G^P = \langle M_a^P, M_b^P, M_c^P \rangle \qquad G^Q = \langle M_a^Q, M_b^Q, M_c^Q \rangle. \tag{12}$$

Since adjacency matrices themselves are also permutation matrices introduced in section 4, G^P and G^Q can be supposed to be subgroups of the symmetric group S_N, so they are finite. Let F_3 be a free group generated by characters a, b and c. We can define the surjective homomorphism

$$\mathcal{M}^P : F_3 \ni \gamma = \gamma_1\gamma_2 \ldots \gamma_m \longmapsto \mathcal{M}^P(\gamma) = M_{\gamma_m}^P M_{\gamma_{m-1}}^P \cdots M_{\gamma_1}^P \in G^P. \tag{13}$$

We denote by $\ker \mathcal{M}^P$ the kernel of the homomorphism $\mathcal{M}^P$, which is the inverse image of $N \times N$ identity matrix E, then G^P is isomorphic to the quotient group $F_3/\ker \mathcal{M}^P$ by the homomorphism theorem. G^Q is also isomorphic to $F_3/\ker \mathcal{M}^Q$ in the same way.

We now assume that $n^P(\gamma) = n^Q(\gamma)$ for any sequence γ, and note that $\mathcal{M}^P(\gamma) = E \Leftrightarrow n_P(\gamma) = N$. Then we have

$$\begin{aligned}\ker \mathcal{M}^P &\equiv \{\gamma \mid \mathcal{M}^P(\gamma) = E\} \\ &= \{\gamma \mid n_P(\gamma) = N\} \\ &= \{\gamma \mid n_Q(\gamma) = N\} \\ &= \{\gamma \mid \mathcal{M}^Q(\gamma) = E\} \equiv \ker \mathcal{M}^Q.\end{aligned} \tag{14}$$

This means that the map

$$\Phi : G^P \ni \mathcal{M}^P(\gamma) \mapsto \mathcal{M}^Q(\gamma) \in G^Q \tag{15}$$

can be defined as an isomorphism between G^P and G^Q. We can suppose identity maps

$$\rho^P : G^P \to GL(N, \boldsymbol{C}) \qquad \rho^Q : G^Q \to GL(N, \boldsymbol{C}) \tag{16}$$

to be linear representations of G^P and G^Q, respectively. Since G^P is isomorphic to G^Q,

$$\rho = \rho^Q \circ \Phi : G^P \ni \mathcal{M}^P(\gamma) \mapsto \mathcal{M}^Q(\gamma) \in GL(N, \boldsymbol{C}) \tag{17}$$

is another linear representation of G^P. Note that $n^P(\gamma)$ and $n^Q(\gamma)$ become the characters of representations ρ^P and ρ respectively, and they are equal by our assumption. Character theory tells us that two representations with the same character are similar [11]. This means that there exists an invertible matrix T such that

$$TM_\gamma^P = M_\gamma^Q T \quad \text{for any } \gamma. \tag{18}$$

This leads to transplantability between P and Q. □

By propositions 1 and 2, we know the equivalence between transplantability and coincidence of the numbers of closed lifts, which are sufficient conditions for isospectrality and isolength spectrality respectively. However if we assume transplantability, which is a stricter condition than isospectrality, then isolength spectrality follows. Conversely, if we assume coincidence of the numbers of closed lifts as a sufficient condition for isolength spectrality, then isospectrality follows. Our results can be summarized as the following schema.

$$\begin{aligned} \text{isospectrality} &\Leftarrow \text{transplantability} \\ &\Leftrightarrow \forall\gamma \quad n^P(\gamma) = n^Q(\gamma) \\ &\Rightarrow \text{isolength spectrality.} \end{aligned}$$

5. List of transplantable pairs for $N \leqslant 13$

As an application of the above propositions, we enumerate isospectral pairs of unfolded domains whose isospectrality can be verified by the transplantation method. The candidate space we have to explore is the set $\mathcal{U}_N$, which consists of all possible edge-coloured graphs without cycles, each of which represents an unfolding rule. The absence of cycles guarantees the unfolded domain to be formed flat for any shaped building block.

The advantage of using the propositions 1 and 2 (and their proof) is that even for the purpose of listing transplantable pairs (or triplets if possible) we do not have to make pairwise comparisons. This is because we can regard each relation $n^P(\gamma) = n^Q(\gamma)$ as a necessary condition for the transplantability of P and Q, and use each relation as a 'filter' to pick up transplantable candidates. More precise steps to find out the transplantable pairs are as follows:

(i) For all edge-coloured graphs P in $\mathcal{U}_N$, compute the character $n^P(\gamma)$ for all possible γ.
(ii) Prepare the spectrum of characters $\{n^P(\gamma)\}$ by ordering the character $n^P(\gamma)$ according to an appropriate rule for γ (lexicographical order of sequence γ, for example).
(iii) Pick up the pair of edge-coloured graphs whose character spectra are the same.

According to the proposition 2, if one finds the identical character spectra, i.e. $\{n^P(\gamma)\} = \{n^Q(\gamma)\}$, they yield the transplantable pairs. One might be afraid that step (i), i.e. enumeration of the character spectrum, seems to require an infinite number of steps, reflecting the fact that γ should run over all possible sequences composed of a, b, c. However, this is not the case, since G^P only forms a finite group as mentioned in the proof of proposition 2.

It is interesting to note that coincidence of the 'ground state', i.e. $n^P(\gamma) = n^Q(\gamma)$ for a null sequence of γ means that the numbers of building blocks forming the unfolded domains are equal, which can be read that the areas of the two domains are equal. Also the condition for the 'first excited state', i.e. $n^P(\gamma) = n^Q(\gamma)$ where $\gamma = a, b$ or c, gives the condition for the length of the boundary to coincide. These geometrical quantities agree with the first and second terms in the Weyl expansion respectively [1, 12, 13]. The rest of the terms are not so straightforwardly interpreted as the first two, but still carry some geometrical information. This can clearly be seen if we limit ourselves particularly to the case where the building block is a triangle whose vertices are labelled by a, b and c. For example, $n^P(\gamma)$ for $\gamma = ab$ represents the number of corners between the sides a and b on the unfolded domain. In the same manner, $n^P(\gamma)$ for $\gamma = abab$ gives how many corners whose angle is twice the angle between the sides a and b appear on the domain and so on. The other patterns not constructed by the repetition of two symbols have no such a clear geometrical meaning.

Applying the above procedure to the edge-coloured graphs up to $N = 13$, we have enumerated all possible transplantable pairs, which are listed in table 2. Figure 5 shows

Table 2. The list of transplantable pairs up to $N \leqslant 13$.

N	$\sharp\mathcal{U}_N$		N	$\sharp\mathcal{U}_N$	
2	3	none	8	450	none
3	3	none	9	1 326	none
4	10	none	10	4 262	none
5	18	none	11	13 566	none
6	57	none	12	44 772	none
7	143	7 pairs	13	148 577	26 pairs

isospectral pairs ever known, which are listed in [9]. Our result completely agrees with the list of known transplantable pairs presented in figure 5 at least up to $N = 13$. Note that all 17 examples given in [9] have been generated based on the Sunada method.

6. Summary and discussion

We have shown that transplantability of eigenfunctions on unfolded domains is equivalent to coincidence of the character spectrum $\{n(\gamma)\}$. The former is a sufficient condition for isospectrality and the latter for isolength spectrality.

As mentioned in the introduction, if the so-called *almost conjugate condition* (5) is satisfied for a pair of subgroups H_1 and H_2, one knows that the quotients M/H_1 and M/H_2 are isospectral, and isolength spectrality of M/H_1 and M/H_2 simultaneously follows. The first counter-example of Kac's original problem has been constructed with this Sunada condition [7]. Once an isospectral pair is obtained based on such a group theoretical argument together with the orbifold construction, one can easily check the isospectrality of the resulting pair using the transplantation method. In this way, at the beginning, the transplantation procedure serves only as a check or demonstration that a given pair of domains is really isospectral. It is uncertain whether or not there exist transplantable pairs of domains which are not constructed based on the Sunada condition, while the Sunada condition is a sufficient condition for transplantability.

However, by expressing an explicit condition (5) for transplantability using adjacency matrices, one notices the similarity between the almost conjugate condition (5) in Sunada's theorem and the condition for existence of the non-permutation matrix T in (9). Even if there exists an invertible matrix T satisfying (9), it is not necessarily true that they give an isospectral pair because they are congruent if T is merely a permutation matrix. Such a situation reminds us of the case where the almost conjugate condition (5) is satisfied but the subgroups H_1 and H_2 are conjugate, which gives only an isometric pair.

Therefore it would become a natural question to inquire about the relation between the Sunada construction of isospectral domains and the transplantation method. The numerical enumeration attempted in section 5 elucidated that at least up to $N = 13$ all the transplantable pairs are those pairs derived by Sunada's method. This suggests the possibility of its equivalence.

The condition $n^P(\gamma) = n^Q(\gamma)$ for all possible sequences of γ, which is equivalent to transplantability of eigenfunctions, not only provides an efficient algorithm to enumerate transplantable pairs of planar domains, but has an interesting 'geometrical' interpretation: Some $n^P(\gamma)$ can straightforwardly be interpreted as the coefficients of Weyl expansion as mentioned in section 5.

The transplantation method, on which we have relied in this paper, is available only to the class of unfolded domains. Whether isospectrality is equivalent to isolength spectrality in more general domains is still an open problem.

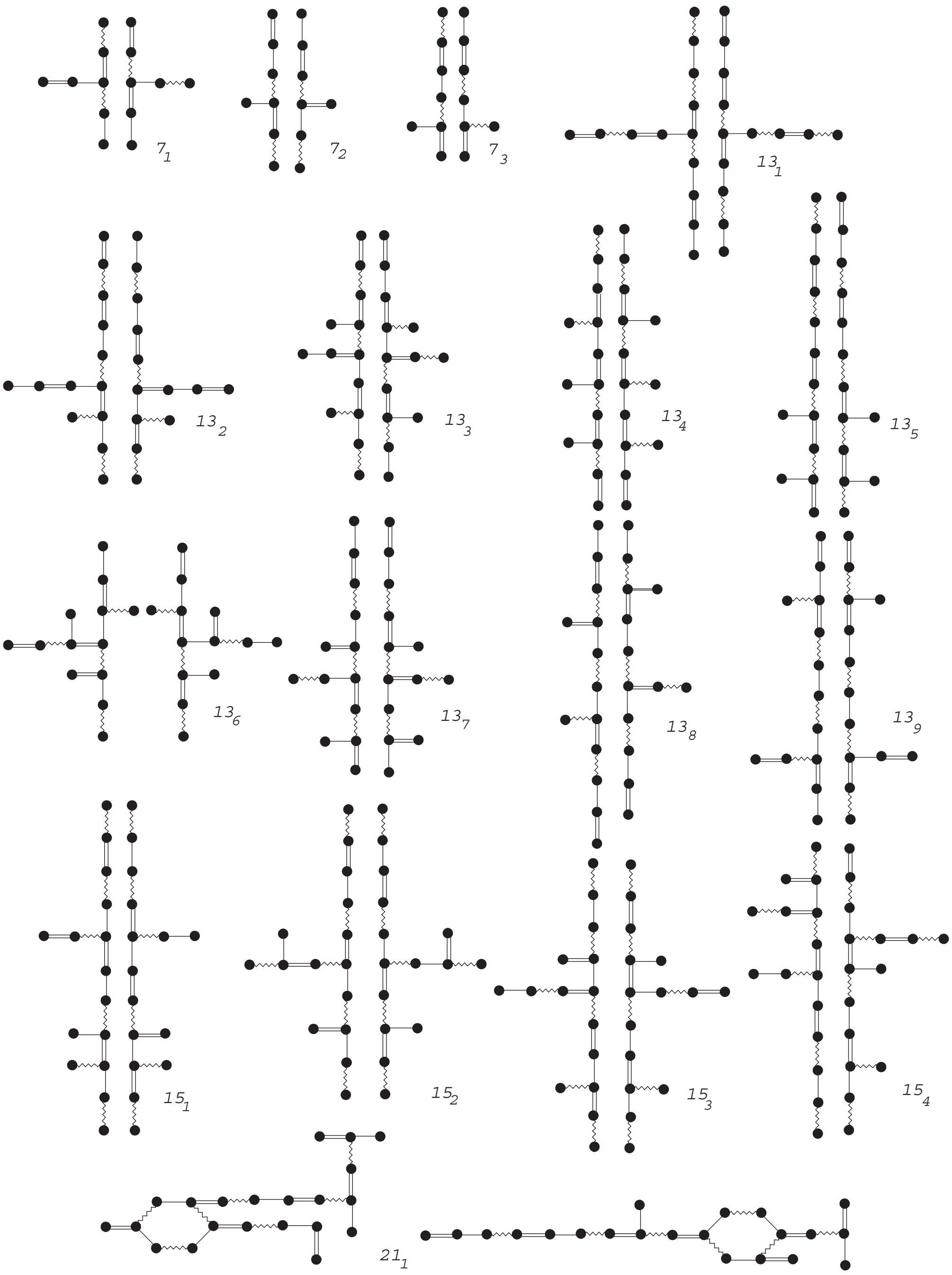

Figure 5. Known isospectral pairs of domains represented as an edge-coloured graph. Three colours of edges are distinguished by the types of line (solid, zig-zag, double), and their loops are omitted. Note that if the colours of edges are permuted the resulting pairs also become isospectral. For example, the pair 7_2 represents three distinguishable pairs of domains for a fixed building block.

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References

[1] Kac M 1966 *Am. Math. Monthly* **73** 1–23
[2] Milnor J 1964 *Proc. Natl Acad. Sci. USA* **51** 542
[3] Sunada T 1985 *Ann. Math.* **121** 169–86
[4] Brooks R 1988 *Am. Math. Monthly* **95** 823–39
[5] Buser P 1986 *Ann. Inst. Fourier, Grenoble* **36** 167–92
[6] Brooks R and Tse R 1987 *Nagoya Math. J.* **36** 823–39
[7] Gordon C, Webb D and Wolpert S 1992 *Invent. Math.* **110** 1–22
[8] Bérard P 1992 *Math. Ann.* **292** 547–59
[9] Buser P, Conway J, Doyle P and Semmler K D 1994 *Int. Math. Res. Not.* **9**
[10] Brooks R 1999 *Contemp. Math.* **231** 25–35
[11] Serre J P 1977 *Linear Representations of Finite Groups* (Berlin: Springer)
[12] Berg M and Srisatkunarajah S 1988 *J. London Math. Soc.* **37** 119–27
[13] Mckean H P Jr and Singer I M 1967 *J. Differ. Geom.* **1** 43–69